"Syncope Implies Chaos" in Walking-stick Maps

Otto E. Rössler

Institute for Physical and Theoretical Chemistry, University of Tübingen,
and Institute for Theoretical Physics, University of Stuttgart, W. Germany

A number of 3-variable chemical and other systems capable of showing 'nonperiodic' oscillations are governed by walking-stick shaped maps as Poincaré cross-sections in state space. The 2-dimensional simple walking-stick diffeomorphism contains the one-dimensional 'single-humped' Li-Yorke map (known to be chaos producing) as a 'degenerate' special case. To prove that chaos is possible also in strictly 2-dimensional walking-stick maps, it suffices to show that a homoclinic point (and hence an infinite number of periodic solutions) is possible in these maps. Such a point occurs in the second iterate at a certain (modest) 'degree of overlap' of the walking-stick map. At a slightly larger degree, a 'nonlinear horseshoe map' is formed in the second iterate. It implies presence of periodic trajectories of all even periodicities (at least) in the walking-stick map. At the same time, two major, formerly disconnected, chaotic subregimes merge into one. Diagnostic criterion: presence of 'syncopes' in an otherwise non-monotone sequence of amplitudes.

1. Introduction

Continuous 3-variable systems showing an apparently nonperiodic oscillatory behavior have been described first in hydrodynamic systems by Lorenz[1], Moore and Spiegel[2], and Cook and Roberts[3]. Later, similarly behaving systems were also found in abstract reaction kinetics[4-8], in abstract neurodynamics[9], and in abstract ecology[10]. Concrete chemical systems displaying the same behavior have been observed since[11, 12].

All these systems can be classified into two major classes[13]: those in which the chaotic behavior is due to the presence of a 'folded' Poincaré map in state space, and those in which it is due to a 'cut' Poincaré map. The systems in[2, 4-7, 9-12] belong to the former category, while those in[1, 3, 8] belong to the latter.

The observation that the former 'zoo' is larger at the time being may reflect the fact that a single second-order nonlinearity suffices for the generation of this type of behavior (see[7] and[14] for examples), while at least two second-order nonlinearities are required for the 'cut' type (see[1] and[13]).

The qualitative behavior of the 'cut' class has been elucidated recently[15, 16], in accordance with earlier results of Lorenz[1]. The underlying Poincaré map[15] has an interesting ('sandwich-like'[13]) structure and is related to a socalled Baker's transformation[17].

In contrast, the qualitative behavior of the simpler, 'folded' class is still less well understood. In the following, the existence of a simple connection will be established between walking-stick maps on the one hand, and the well-known dynamical notions of homoclinic point and horseshoe map on the other hand.

2. Some Historical Remarks

Systems that are governed by a folded Poincaré map in state space are actually known for a rather long time. Khaikin's 'universal circuit'[18] is a prototypic example; its non-periodic capabilities have nonetheless been overlooked for a long time[4]. A differential equation of the same class was also considered by Smale[19], who expressed the "feeling . . . that one can expect the qualitative behavior of this differential equation in 3-space to be rather complicated" and suggested that a study of its Poincaré map "should be a relevant, interesting and challenging problem", but gave no clues as to the form of this map. Moore and Spiegel[2], in turn, did observe non-periodic oscillations in their system (which again is of related type), but did not look at its Poincaré map. Kopell[20], in studying a non-linear 3-variable control system, the Danziger-Elmergreen oscillator[21], stressed that this system should be capable of non-trivial recurrent oscillations in the sense of Birkhoff[22]. This system indeed shows chaos if a slight modification of the piece-wise linear original feed-back term is allowed[23].

The first example of a chaotic trajectorial flow governed by a walking-stick map seems to be the

'3-dimensional mincer'[24] or 'blender'[4] of abstract reaction kinetics. At that time it seemed[4] that the underlying folded map was simply identical with Smale's well-known horseshoe map[25] — which is not the case, however[26, 13]. Hénon[27] found a folded regime in numerical studies of cross-sections through the trajectorial flow of the Lorenz equation at a certain set of parameter values. The same finding was independently made in[28], where the corresponding map was classified as a (non-linear) 'horseshoe map'. Ruelle[29] has also reported on a folded map, as cited by Hénon[30]. Hénon's own folded map[30], involving a rotation by exactly 90 degrees, is not assumed to be realizable by simple continuous flows. The same holds true for Oster and Guckenheimer's piece-wise linear map[31].

3. The Walking-Stick Map

In Fig. 1, the walking-stick map[13] is depicted: a more or less rectangular disk is mapped onto part of itself in such a way that it is first 'elongated' and 'thinned' and then 'folded over' (at one end) before being put back into itself[4, 5, 7]. The arrowheads in the figure are to facilitate identification. The map is assumed to be a diffeomorphism (that is, a differentiable one-to-one and onto map whose inverse is also differentiable[32]). This is because the map is assumed to be 'suspended' (cf.[25]) by a trajectorial flow generated by a set of ordinary differential equations. Moreover, it is assumed that the map is 'simple', meaning that unnecessary distortions (apart from those directly related to the elongating, compressing and folding procedure) are either absent or small. Specifically, the 'elongation' is to be positive everywhere, so that there are no pieces that are 'horizontally compressed' during the process, except (if necessary for the region near the prospective 'knee'. (This 'monotonicity assumption' seems to be justified in realistic differential systems like that in[7].)

As seen in Fig. 1, the image of the disk always lies in its interior. It is therefore possible to
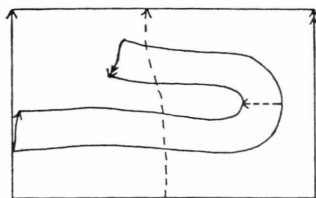

Fig. 1. The walking-stick map (see text).

'nibble away' from the sides of the disk, until no 'superfluous parts' (to which no points return) are left. This leads to the pictures of Fig. 2 which comprise the only interesting portion of the map. There are two possibilities as shown, depending on the 'degree of overlap' of the original map: Fig. 2 a applies to the case where the overlap of the original
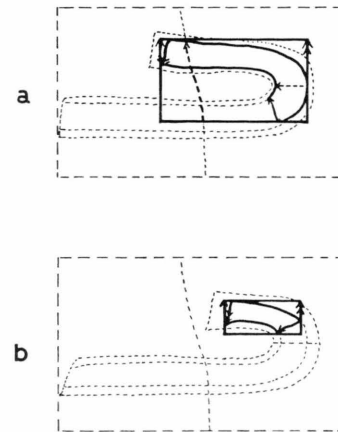

Fig. 2. The walking-stick map sensu stricto (see text). **a** is the non-trivial case.

map goes beyond the dashed line of Fig. 1, which is the prospective 'knee line' (also dashed). Figure 2 b is obtained if this is not the case. Only the non-trivial case of Fig. 2 a will be considered in the following. It will be called the 'walking-stick map in the restricted (or proper) sense'.

4. There is Exactly One Fixed Point in the Walking-Stick Map

The left, and prospective 'lower', part of the walking-stick map (Fig. 2 a) cannot contain a fixed point since it does not map upon itself. The right-hand, and 'upper', portion of the map contains at least one fixed point, for part of it maps on itself. The fact that something like a 'monotone expansion' applies to horizontal directions and something like a 'monotone contraction' to vertical directions, suggests that there may be a single fixed point of saddle character.

This conjecture is confirmed on closer inspection. A representative case is depicted in Figure 3. Again, a 'cutting-off' procedure with respect to those parts of the map that do not return, is applied. In Fig. 3 a, the iterative cutting process is confined to the horizontal direction. It is bound to come to a standstill
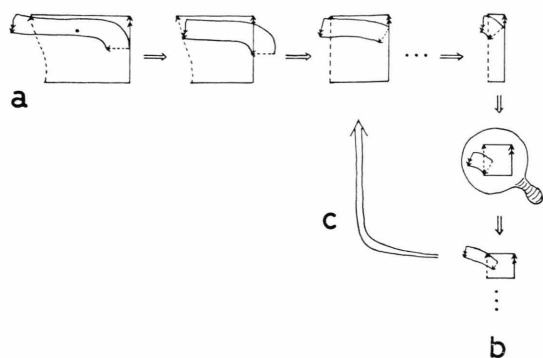
Fig. 3. An iterative narrowing-down procedure applied to the right-hand part of the walking-stick map. Dot = prospective fixed point. **a**: horizontal narrowing-down; **b**: vertical narrowing-down; **c**: horizontal narrowing-down again (see text).

(unless the 'main axes' of the original domain and its picture are strictly parallel, which is non-generic). As shown in Fig. 3 b, the same cutting method can now be performed in the vertical direction, until another standstill is reached. Thereafter, the horizontal procedure can be applied again, and so forth. In Fig. 3 c, a 'short-cut' is shown. Within this cycle, the situation becomes more and more 'identical' (except for the shrinking size) between one

round of iteration and the next. This assures convergence (and, ultimately, linearity).

The resulting single fixed point has the properties sketched in Fig. 4 a, if it is located in a sufficiently 'horizontal' segment of the first iterate. The fixed point then is a saddle point with the unusual (and in continuous 2-dimensional dynamical systems impossible) property that the points to the left and to the right of each eigenvector (seeparatrix) are oscillating around the saddle (as indicated). If there is no 'horizontal net elongation' (Fig. 4 b), the fixed point is a stable node (again with an orientation reversal occurring in the directions of both of its eigenvectors). If the fixed point is located in a more 'vertical' segment, it is an unstable focus (supposed there is sufficient elongation): Figure 4 c. Otherwise it is a stable focus. These are the only generic possibilities. (Notice that, according to the Hartman-Grobman theorem, the local neighborhood of hyperbolic fixed points — like those described — is a linear automorphism, namely, the local derivative; see [25].)

5. Second Iterate Contains Two Walking-Stick Maps

The second iterate of a walking-stick map has the form seen in Figure 5 b. The fixed point inherited from the first iterate (Fig. 5 a) has also been en-
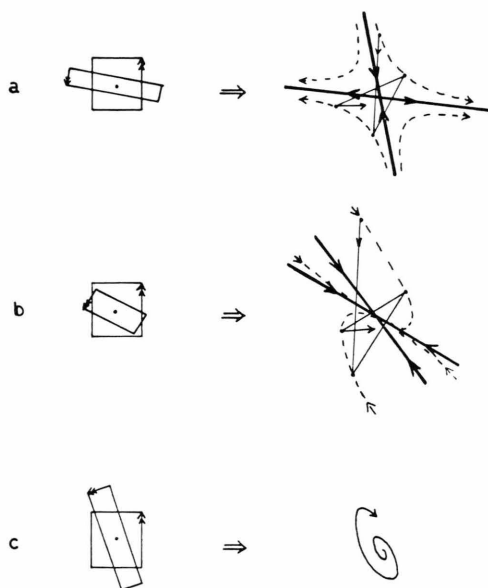
Fig. 4. The three types of fixed points possible in the map of Figure 3. **a**: unstable saddle-like fixed point; **b**: stable node-like fixed point; **c**: unstable focus. In **a** and **b**, some consecutive points have been connected by arrows on either side of the fixed point in order to demonstrate the oscillatory orientation reversal (see text).

Fig. 5. The second iterate (**b**) of a walking-stick map (**a**) contains two walking-stick maps (**c, d**).

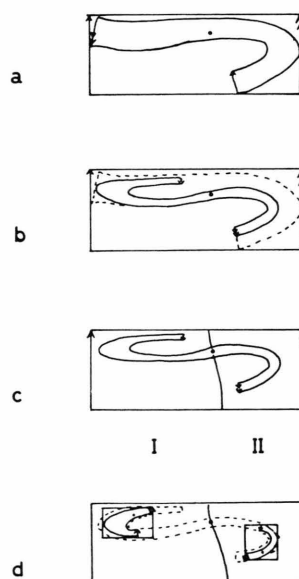tered. While the fixed point of Fig. 5 a is the 'oscillatory' saddle of Fig. 4 a (at the assumed degree of overlap and 'flatness' of the walking-stick), its image in Fig. 5 b lacks the oscillatory component since only every second passage through the original map is counted. It thus is an ordinary saddle, attracting in a roughly vertical direction and repelling in a roughly horizontal direction.

As a consequence, Fig. 5 b contains two sub-maps which each have the very structure of Figure 1. This is emphasized in Figure 5 c. Figure 5 d shows that again two 'proper' walking-stick maps may be formed, in dependence on the degree of overlap of the original map.

Obviously, the overlap in the two new (proper) walking-stick maps is generically either smaller or larger than that of the original proper map.

6. Some Quantitative Implications

In the general case, no quantitative predictions as to the ratio of overlaps can be made, especially as far as higher iterates are concerned. The reason for this (as we shall see, only partial) drawback is that the position of the fixed point in the respective next walking-stick map depends not only on the relative positions of fixed point and overlapping edge of the last map, but as well on other parts of that map, about which nothing is known in general.

Nonetheless, the following statement can be made: if the 'edge' of the overlapping portion of the original map comes 'sufficiently close' to the level of the fixed point, the next map will show an even greater overlap with respect to its own fixed point. Thus, there is a 'threshold of overlap' beyond which an 'amplification of overlaps' occurs. It is valid for one round of iteration.

7. Sufficient Overlap Implies Homoclinic Point for the Second Iterate

The last finding of a possible 'amplification' of overlap includes as a special case the situation depicted in Figure 6. Here the overlap of the second iterate is 'total'. This occurs whenever the level of the fixed point in the original map is actually surpassed by the edge of the handle of the walking-stick. This still corresponds to a quite 'moderate' overlap in terms of the original map.
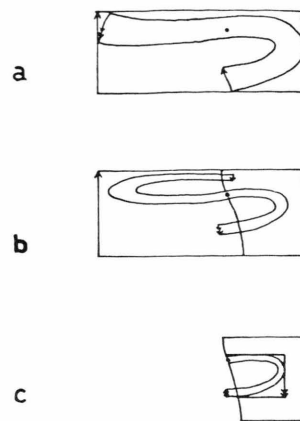

Fig. 6. Formation of 'total overlap' in the second iterate (see text).

One of the new maps formed is shown once more in Fig. 6 c, again with unnecessary parts omitted (box). The map in the box is closely related to Smale's horseshoe map [25]. Only the 'linearity condition' assumed and stressed by Smale [25] is not fulfilled. This justifies the name 'non-linear horseshoe map' for the new map.

The most remarkable property of the new map is illustrated in Figure 7. It is the formation of a transversal homoclinic point (H in Figure 7). Such a point consists of the intersection between the stable
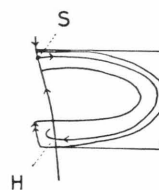

Fig. 7. The 'non-linear horseshoe map', with saddle point (S) and transversal homoclinic point (H) entered (see text).

and the unstable manifold of a saddle point (S). (Notice that the existence of those manifolds is assured by the stable manifold theorem, e. g. [25]; their location within the map follows directly from the construction of the map.)

Poincaré [33] first detected the possibility that in surface transformations, the stable and unstable separatrix of a saddle-type fixed point may intersect. He also recognized that a single such crossing implies an infinite number of crossings under subsequent iterations (so that a non-periodic trajectory is formed). Poincaré [33] incidentally also stressed that such points and trajectories should readily form

in non-linear 3-variable differential systems (cited after [34]).

Later, Birkhoff [35] gave a simple geometric proof that in the neighborhood of every homoclinic point, an infinite number of periodic solutions exist. Smale [36] showed, specifically, that every transversal homoclinic point determines the formation of a 'shift automorphism', so that presence of periodic solutions of all integer periodicities is implied, for some iterate of the map.

8. Walking-Stick Map of Sufficient Overlap Implies Chaos of all Even Periodicities

In the present case, chaos of all integer periodicities is implied by the map that contains H (the nonlinear horseshoe map), so that there is chaos of all even periodicities in the original map. A sufficient condition is that the overlap of the non-linear horseshoe map be great enough that the 'knee' of the map is being 'cut out in total' at the next iteration. This condition is illustrated in Figure 8.

Fig. 8. Sufficient overlap for chaos with all integer periodicities in the non-linear horseshoe map.

It is now possible to proceed with the same fixed-point argument as used in Section 4. To begin, the new fixed point for the first iterate of the non-linear horseshoe map (corresponding to a period-2 solution of the basic walking-stick map) is formed in the same manner as that in Fig. 3; the resulting fixed pointhas been entered into Fig. 8 (approximate position). The second iterate of the non-linear horseshoe map also contains a new fixed point (corresponding to period-4 of the basic map), see Figure 9. In Fig. 9 a, the whole map is shown. A (half-)portion is repeated in Figure 9 b. Only this portion need be considered: a whole region (namely, that located between the two 'vertical' arrows) has been folded back over itself (and pulled through itself) in such a way that the image of the right-hand border finally lies to the left of the left-hand border. This means that the iteration process used in Fig. 3 can be applied again. The first two steps are shown in Figs. 9 c and 9 d; from here, Fig. 3 (third picture of Fig. 3 a) takes over. Hence there is a (saddle-like) fixed point of
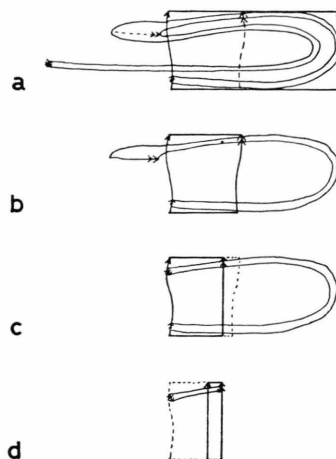

Fig. 9. Formation of a non-trivial period-2 fixed point in the second iterate of the non-linear horseshoe map of Figure 8 (see text).

period-4 in the original map. Similarly, Fig. 10 shows a half-portion of the third iterate. The lower part of this picture has the same structure as the relevant (middle and left-hand) parts of Fig. 9 b, so that Fig. 9 d is obtained again in two steps. This leads to a fixed point of period 6 in the basic map. Figure 11, finally, shows a half-portion of the fourth iterate, where the situation is again the same (yielding period-8 in the basic map); and so forth.

Fig. 10. Period-3 fixed point in the third iterate.

Fig. 11. Period-4 fixed point in fourth iterate (and so forth).

Thus, there is a degree of overlap in walking-stick maps ('a bit beyond the level of the fixed point') which determines presence of periodic solutions of (at least) all even periodicities.

9. Connected Chaos Occurs Last

The finding of Section 6 (amplification of overlaps beyond a certain threshold) implies that the same mechanism which leads to the formation of a transversal homoclinic point ('total overlap') in the

second iterate, will be fulfilled for some higher iterate first, if a continually increasing overlap is assumed. Especially the two walking-stick maps generated in the fourth iterate by one of the walking-stick maps of the second iterate will show total overlap earlier than those of the second iterate do.

This means that two (indeed, many) disconnected chaotic regimes appear first under a gradual increase of overlap. They then merge in a pairwise manner until, finally, only a single chaotic regime remains. Thus, 'connected chaos' (defined by the absence of major disconnected regimes) occurs last [37].

10. Syncope Implies Chaos

As long as there are two (or more) independent chaotic subregimes in a walking-stick map system, this is reflected by an 'amplitude gap' in the otherwise irregular oscillations: certain values of 'horizontal' amplitude are never observed.
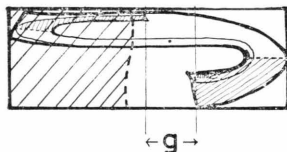

Fig. 12. Formation of 'amplitude gap' at a degree of overlap that is sufficient for chaos to occur in the two submaps of Figure 5 d. g = medium strip (see text).

The formation of the amplitude gap is illustrated in Figure 12. It is readily verified that the shaded parts (corresponding to the 'lower' portion of the first iterate) will never map onto a certain medium strip of the map in which also the fixed point is contained, no matter what the number of iterations. Since the strip belongs to the prospective 'upper' part of the map (which is not folding but just expanding; see Section 4), all points starting in this region will soon end up in the shaded parts. Hence the formation of a 'gap' in the horizontal amplitudes. The gap can be expected to be reflected in most continuous observations of a physical system that is governed by this type of a map.

A closing of the gap is equivalent to the overlap reaching the level of the fixed point. (More precisely, the gap is defined by the distance of the edge of the walking-stick's handle to the ω-separatrix of the saddle; it is 'closed' when the homoclinic point of Fig. 7 appears.) A diagnostic criterion – if the walking-stick map is sufficiently 'flat' – for the

absence of the gap, and hence a fortiori for the presence of chaos, is the absence of strict non-monotonicity between subsequent 'horizontal' amplitudes.
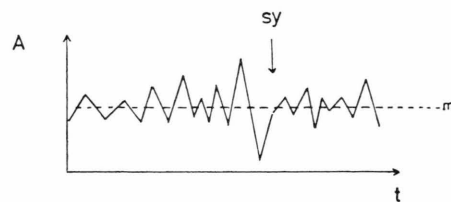

Fig. 13. 'Syncope' is an otherwise non-monotone temporal sequence. A is assumed to be monotonically related to the main (that is, horizontal) amplitude of a walking-stick map. Only subsequent amplitudes (maxima) are shown. m = value about which the amplitude mostly oscillates; sy = syncopic event; t = time.

This is easy to observe (see Fig. 13): an exception to strict non-monotonicity imposes as a sort of 'syncope'. (For example, two steps up in a row, or two steps down in a row.) The criterion "syncope implies chaos" may – in view of the fact that walking-stick map chaos constitutes a frequently occurring type (see Introduction) – perhaps even acquire practical significance.

11. Discussion

The simplest chaos producing diffeomorphic map, the walking-stick map, proved easily analyzable. After a few simple steps, the Poincaré-Birkhoff-Smale theory of homoclinic points could be applied. In addition, the distinction of 'disconnected' vs. 'connected' chaos became possible. Connected chaos has the asset of being easily recognizable (by 'syncopes').

It should be pointed out now what has not been shown. In the first line, the absence of a periodic attractor (limit cycle) of large periodicity has not been shown. The probability is great that such an attractor is almost always present [26, 13], meaning that all observed non-periodic time segments of oscillation are either transients or parts of long cycles, if complete absence of perturbations is presupposed. In this respect, walking-stick map chaotic systems are 'inferior' to sandwich map chaotic systems; for sandwich map systems generically possess a strange attractor [15, 16] (that is, a non-periodic limit set). Walking-stick map systems can only be said to possess a strange quasi-attractor [13], if indeed almost every at first irregular trajectory is eventually (at some un-

predictable moment in time) 'caught' by a periodic attractor. This mathematically interesting distinction is, nonetheless, irrelevant from the point of view of applications (cf. [38]).

A second drawback is that a direct analogue to the Li-Yorke theorem [39], "period 3 implies chaos", has not been given. The one-dimensional Lorenz-Li-Yorke map, if 'single-humped', is a 'degenerate' special case to the here considered map [13]. Thus, "syncope implies chaos" also for the one-dimensional case. As a diagnostic criterion, this theorem is even finer than the period 3 theorem. But it does not, of course, prove the presence of periodic solutions of all periodicities, both even and odd, as the Li-Yorke criterion does. Moreover, the "period 6 implies chaos" criterion, which is implicit in the Li-Yorke theorem, is somewhat finer in turn than the "syncope" criterion (although the latter is easier to apply).

Validity of the Li-Yorke theorem ("period 3 implies periodic solutions of all integer periodicities")

remains to be demonstrated for 2-dimensional walking-stick maps. A combination of Li and Yorke's proof [39] with the 2-dimensional cutting procedure used above seems promising.

There is a third open question, both in the 2-dimensional and in the one-dimensional case: the 'threshold for chaos' problem (What overlap is just sufficient for chaos to arise for the first time in some higher iterate?). In [40], the Li-Yorke criterion was efficiently used in a numerical investigation into a specific one-dimensional map – yielding a certain numerical value for the overlap in this particular case. It would be interesting to compare the power of the Li-Yorke theorem in this respect with that of the syncope criterion (by applying both to higher and higher iterates) [41].

To conclude, the conjecture [13] that walking-stick maps of sufficient overlap determine chaos also in the 'non-degenerate' case, has been confirmed.

[1] E. N. Lorenz, J. Atmos. Sci. **20**, 130 [1963].
[2] D. W. Moore and E. A. Spiegel, Astrophys. J. **143**, 871 [1966].
[3] A. E. Cook and P. H. Roberts, Proc. Camb. Phil. Soc. **68**, 547 [1970].
[4] O. E. Rössler, Z. Naturforsch. **31 a**, 259 [1976].
[5] O. E. Rössler, Bull. Math. Biol. **39**, 275 [1977].
[6] O. E. Rössler, Z. Naturforsch. **31 a**, 1168 [1976].
[7] O. E. Rössler, Phys. Letters **57 A**, 397 [1976].
[8] O. E. Rössler, Strange Attractor in a 3-variable Abstract Reaction System. (In Preparation.)
[9] R. FitzHugh, Personal Communication [1976].
[10] M. Gilpin, Personal Communication [1977].
[11] L. F. Olsen and H. Degn, Chaos in an Enzyme Reaction. (Preprint.)
[12] O. E. Rössler and K. Wegmann, Chaos in the Zhabotinsky Reaction. (In Preparation.)
[13] O. E. Rössler, Z. Naturforsch. **31 a**, 1664 [1976].
[14] O. E. Rössler, Toroidal Oscillations and Chaos in Two Simple Differential Equations. (In Preparation.)
[15] J. Guckenheimer, A Strange, Strange Attractor. In: "The Hopf Bifurcation" (J. E. Marsden and M. McCracken, eds.), p. 368, Springer-Verlag, New York 1976.
[16] R. F. Williams, The Structure of Lorenz Attractors. (Preprint.)
[17] Cited after V. I. Arnold and A. Avez, Ergodic Problems in Classical Mechanics. Benjamin, New York 1970; first French ed. 1966.
[18] S. E. Khaikin, Zh. Prikl. Fiz. **7**, 21 [1930]; A. A. Andronov, S. E. Khaikin, and A. A. Vitt, Theory of Oscillators, p. 725 pp. Pergamon, New York 1966.
[19] S. Smale, J. Differential Geometry **7**, 193 [1972].
[20] N. Kopell, Bull. Math. Biol. **35**, 689 [1973].
[21] L. Danziger and G. L. Elmergreen, Bull. Math. Biol. **18**, 1 [1956].
[22] G. D. Birkhoff, Bull. Soc. Math. d. France **40**, 305 [1912].
[23] O. E. Rössler, Walking-stick Map Chaos in an Equation of Danziger-Elmergreen Type. (In Preparation.)
[24] O. E. Rössler, Unpublished [1975].

[25] S. Smale, Bull. Amer. Math. Soc. **73**, 747 [1967].
[26] S. Smale, Personal Communication [1976].
[27] M. Hénon and Y. Pomeau, Lecture Notes Math. **565**, 29 [1976].
[28] O. E. Rössler, Horseshoe Map Chaos in the Lorenz Equation. Phys. Letters **60 A**, 392 [1977].
[29] D. Ruelle, Report at the Conference on "Quantum Dynamics Models and Mathematics" in Bielefeld, September 1975 (cited after [30]).
[30] M. Hénon, Commun. Math. Phys. **50**, 69 [1976].
[31] J. Guckenheimer, G. Oster, and A. Ipatchki, The Dynamics of Density Dependent Population Models. (Preprint.)
[32] See, for example, V. Guillemin and A. Pollack, Differential Topology, Prentice-Hall, Englewood-Cliffs 1974.
[33] H. Poincaré, Méthodes Nouvelles de la Mécanique Céleste. Paris 1899; reprint: Dover, New York 1957.
[34] S. Lefschetz, Geometric Differential Equations: Recent Past and Proximate Future. In: „Differential Equations and Dynamical System" (J. K. Hale and J. Lasalle, eds.), p. 1, Academic Press, New York 1967.
[35] G. D. Birkhoff, Acta Math. **50**, 359 [1927].
[36] S. Smale, Diffeomorphisms with Many Periodic Points. In: "Differential and Combinatorial Topology", p. 63, Princeton University Press, Princeton, N. J. 1965.
[37] This result is confirmed, for a one-dimensional special case, in an independent study by Grossmann and Thomae: chaos there begins with an infinite number of chaotic sub-regimes, with subsequent treelike convergence toward a single regime when the overlap is increased further. See S. Thomae, Diploma Thesis, Physics Department, University of Marburg 1977.
[38] R. May, Nature London **261**, 459 [1976].
[39] T. Y. Li and J. A. Yorke, Amer. Math. Monthly **82**, 985 [1975].
[40] F. C. Hoppensteadt and J. M. Hyman, Periodic Solutions of a Logistic Difference Equation. (Preprint.)
[41] The two thresholds coincide numerically in the logistic difference equation. (S. Thomae, Personal Communication, 1977.)